

# Approximation properties of the $q$ -Balázs-Szabados operators in the case $q \geq 1$

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## Abstract

This paper deals with approximation properties of the newly defined  $q$ -generalization of the Balázs-Szabados operators in the case  $q \geq 1$ . Quantitative estimates of the convergence and Voronovskaja type theorem are given. In particular, it is proved that the rate of approximation by the  $q$ -Balázs-Szabados ( $q > 1$ ) is of order  $q^{-n}$  versus  $1/n$  for the classical Balázs-Szabados ( $q = 1$ ) operators. The results are new even for the classical case  $q = 1$ .

## 1 Introduction

The goal of the paper is to define a  $q$ -analogue and study approximation properties for the rational complex Balázs-Szabados operators given by

$$R_n(f; z) = \frac{1}{(1 + a_n z)^n} \sum_{k=0}^n f\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n z)^k,$$

where  $a_n = n^{\beta-1}$ ,  $b_n = n^\beta$ ,  $0 < \beta \leq \frac{2}{3}$ ,  $n \in \mathbb{N}$  and  $z \in \mathbb{C}$ ,  $z \neq -\frac{1}{a_n}$ .

In the real form rational operators were introduced and studied in Balázs [1] and Balázs-Szabados [2]. Totik [7] settled the saturation properties of  $R_n(f)$ . Further studies on these operators in the case of real variable can be found in the paper Abel-Della Vecchia [3]. They studied the complete asymptotic expansion for operators  $R_n(f)$  as  $n \rightarrow \infty$ . Approximation properties of the complex Balázs-Szabados operators were studied in Gal [5]. The  $q$ -analogue of these operator was given by Doğru who investigated statistical approximation properties of  $q$ -Balázs-Szabados operators [4]. The approximation properties of the complex  $q$ -Balázs-Szabados operators is studied in [6].

We introduce some notations and definitions of  $q$ -calculus, see [9], [?]. Let  $q > 0$ . For any  $n \in \mathbb{N} \cup \{0\}$ , the  $q$ -integer  $[n]_q$  is defined by

$$[n]_q := \begin{cases} (1 - q^n) / (1 - q), & q \neq 1 \\ n, & q = 1 \end{cases}, \quad [0]_q := 0;$$

and the  $q$ -factorial  $[n]_q!$  by

$$[n]_q! := [1]_q [2]_q \cdots [n]_q, \quad [0]_q! := 1.$$

For integers  $0 \leq k \leq n$ , the  $q$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For fixed  $q > 0$ ,  $q \neq 1$ , we denote the  $q$ -derivative  $D_q f(z)$  of  $f$  by

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Let us introduce a  $q$ -Balázs-Szabados operator.

**Definition 1** Let  $q > 0$ . For  $f : [0, \infty) \rightarrow \mathbb{R}$  we define the Balázs-Szabados operator based on the  $q$ -integers as follows.

$$R_{n,q}(f; x) = \frac{1}{(1 + a_n x)^n} \sum_{k=0}^n f\left(\frac{[k]_q}{b_n}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q (a_n x)^k \prod_{s=0}^{n-k-1} \left(1 + (1-q)[s]_q a_n x\right), \quad (1)$$

where  $a_n = [n]_q^{\beta-1}$ ,  $b_n = [n]_q^\beta$ ,  $0 < \beta \leq \frac{2}{3}$ ,  $n \in \mathbb{N}$  and  $x \neq -\frac{1}{a_n}$ .

In the case  $q = 1$  these polynomials coincide with the classical ones. For  $q \neq 1$  one gets a new class of polynomials having interesting properties. It should be mentioned that in the case  $q \in (0, 1]$   $q$ -Balázs-Szabados operators generate positive linear operators  $R_{n,q} : f \rightarrow R_{n,q}(f; x)$ . In the case  $q > 1$  positivity fails, however, the results of this paper show that in this case approximating properties of  $q$ -Balázs-Szabados operators may be better than the case  $0 < q \leq 1$ .

Throughout this paper, let  $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$  denote the disk of radius  $R$  centered at 0. Moreover, it is assumed that  $f(z) = \sum_{m=0}^{\infty} c_m z^m$ , for  $z \in \mathbb{D}_R$ .

Assuming  $f : \mathbb{D}_R \cup [R, +\infty) \rightarrow \mathbb{C}$  and simply replacing  $x$  by  $z$  in (1) we obtain the complex form of the  $q$ -Balázs-Szabados operator:

$$R_{n,q}(f; z) = \frac{1}{(1 + a_n z)^n} \sum_{k=0}^n f\left(\frac{[k]_q}{b_n}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q (a_n z)^k \prod_{s=0}^{n-k-1} \left(1 + (1-q)[s]_q a_n z\right),$$

where again  $a_n = [n]_q^{\beta-1}$ ,  $b_n = [n]_q^\beta$ ,  $0 < \beta \leq \frac{2}{3}$ ,  $n \in \mathbb{N}$ ,  $z \in \mathbb{C}$  and  $z \neq -\frac{1}{a_n}$ .

**Remark 2** The complex operators  $R_{n,q}(f; z)$  are well defined and analytic for all  $n \geq n_0$  and  $|z| \leq r < [n_0]_q^{1-\beta}$ . Indeed, in this case we easily obtain that  $z \neq -\frac{1}{a_n}$ , for all  $|z| \leq r < [n_0]_q^{1-\beta}$  and  $n \geq n_0$ , which implies that  $1/(1 + a_n z)^n$  is analytic.

**Remark 3** There exists a close connection between  $R_{n,q}(f; z)$  and the complex  $q$ -Bernstein polynomials given by

$$B_{n,q}(f; z) = \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{s=0}^{n-k-1} (1 - q^s z).$$

Indeed, denoting  $F_n(z) = f\left(\frac{[n]_q}{b_n} z\right)$ , we easily get

$$R_{n,q}(f; z) = B_{n,q}\left(F_n; \frac{a_n z}{1 + a_n z}\right),$$

valid for all  $n \geq n_0$  and  $|z| \leq r < [n_0]_q^{1-\beta}$ . This connection will be essential in our reasonings. For monomials  $f(z) = e_m(z) = z^m$  it can be written as follows:

$$R_{n,q}(e_m; z) = [n]_q^{m(1-\beta)} B_{n,q}\left(e_m; \frac{a_n z}{1 + a_n z}\right).$$

**Remark 4** The lack of positivity makes the investigation of convergence in the case  $q > 1$  essentially more difficult than for  $0 < q < 1$ . Notice that, the complex  $q$ -Bernstein type operators in the case  $q > 1$  systematically are studied in [10], [11], [12], [13], [14], and [15].

**Remark 5** Approximation properties of the complex Balázs-Szabados operators are studied in [5]. Notice that unlike to [5] the growth conditions of exponential-type on  $f$  is omitted. The only condition imposed to  $f$  is to be uniformly continuous and bounded on  $[0, +\infty)$ . Therefore our results are new even for the classical Balázs-Szabados operators.

**Theorem 6** Let  $n_0 \geq 2$ ,  $0 < \beta \leq \frac{2}{3}$ . Assume that  $f : \mathbb{D}_R \cup [R, +\infty) \rightarrow \mathbb{C}$  is uniformly continuous and bounded on  $[0, +\infty)$ , is analytic in  $\mathbb{D}_R$ . Then

$$|R_{n,q}(f; z) - f(z)| \leq \frac{1}{[n]_q^\beta} \sum_{m=2}^{\infty} |c_m| m(m-1) (4q^2 r)^m + \frac{2r}{[n]_q^{1-\beta}} \sum_{m=1}^{\infty} |c_m| (2r)^m,$$

$$q \geq 1, \quad \frac{1}{2} < r < \frac{R}{4q^2} \leq \frac{1}{2} [n_0]_q^{1-\beta}.$$

In [7], Totik settled the saturation properties of  $R_n$ . Among other things he proved the Voronovskaja-type result for  $0 < \beta \leq \frac{1}{2}$ ,  $\beta \geq \frac{2}{3}$ . The complete asymptotic expansion for  $R_n$  is given by Abel and Della Vecchia [3].

Next, we study Voronovskaja type formulas of the  $q$ -Balázs-Szabados operators of a function  $f$  analytic in the disc  $\mathbb{D}_R$ . In order to formulate Voronovskaja type theorem we define the following function

$$L_q^\beta(f; z) := \begin{cases} \frac{D_q f(z) - f'(z)}{q-1}, & \text{if } |z| < R/q, R > q > 1, 0 < \beta < \frac{1}{2}, \\ -z^2 f'(z) + \frac{D_q f(z) - f'(z)}{q-1}, & \text{if } |z| < R/q, R > q > 1, \beta = \frac{1}{2}, \\ -z^2 f'(z), & \text{if } |z| < R/q, R > q > 1, \frac{1}{2} < \beta < 1, \end{cases} \quad (2)$$

and for  $q = 1$ ,

$$L_q^\beta(f; z) := \begin{cases} \frac{z}{2} f''(z), & \text{if } |z| < R, 0 < \beta < \frac{1}{2}, \\ -z^2 f'(z) + \frac{z}{2} f''(z), & \text{if } |z| < R, \beta = \frac{1}{2}, \\ -z^2 f'(z), & \text{if } |z| < R, \frac{1}{2} < \beta < 1. \end{cases}$$

In the case of complex variable, the qualitative Voronovskaja-type result for  $R_n$  is proved by Gal [5]. Note that the case  $\beta = \frac{1}{2}$  remained open in [5]. We prove the following quantitative Voronovskaja type theorem for  $R_{n,q}$ , which covers the case  $\beta = \frac{1}{2}$ . Moreover, our results are new for the classical Balázs-Szabados operators ( $q = 1$ ).

**Theorem 7** Let  $n_0 \geq 2$ ,  $0 < \beta \leq \frac{2}{3}$ . Assume that  $f : \mathbb{D}_R \cup [R, +\infty) \rightarrow \mathbb{C}$  is uniformly continuous and bounded on  $[0, +\infty)$ , is analytic in  $\mathbb{D}_R$ . Then

(i) For  $0 < \beta < \frac{1}{2}$ ,  $\frac{1}{2} < r < \frac{R}{\max(4q, 2q^2)} < R \leq \frac{1}{2} [n_0]_q^{1-\beta}$ ,  $|z| \leq r$ , we have

$$\left| R_{n,q}(f; z) - f(z) - \frac{1}{[n]_q^\beta} \frac{z(D_q f(z) - f'(z))}{q-1} \right|$$

$$\leq \frac{4}{[n]_q^{2\beta}} \sum_{m=0}^{\infty} |c_m| (m-2) (4qr)^{m-2} + \frac{4}{[n]_q^{1-\beta}} \sum_{m=0}^{\infty} |c_m| m(m-1) (2q^2 r)^{m+1};$$

(ii) For  $\frac{1}{2} < \beta \leq \frac{2}{3}$ ,  $\frac{1}{2} < r < \frac{R}{4q} < R \leq \frac{1}{2} [n_0]_q^{1-\beta}$ ,  $|z| \leq r$ , we have

$$\left| R_{n,q}(f; z) - f(z) + \frac{1}{[n]_q^{1-\beta}} z^2 f'(z) \right| \leq \frac{6}{[n]_q^\beta} \sum_{m=0}^{\infty} |c_m| m(m-1) (4qr)^m;$$

(iii) For  $\beta = \frac{1}{2}$ ,  $\frac{1}{2} < r < \frac{R}{4q^2} < R \leq \frac{1}{2} [n_0]_q^{1-\beta}$ ,  $|z| \leq r$ , we have

$$\begin{aligned} & \left| R_{n,q}(f; z) - f(z) + \frac{1}{\sqrt{[n]_q}} z^2 f'(z) - \frac{1}{\sqrt{[n]_q}} \frac{z(D_q f(z) - f'(z))}{q-1} \right| \\ & \leq \frac{9}{[n]_q} \sum_{m=0}^{\infty} |c_m| m^2 (m-1)^2 (4q^2 r)^m. \end{aligned}$$

By using the above Voronovskaja's theorem, we will obtain the exact order in approximation by the complex  $q$ -Balázs-Szabados operators. In this sense, we present the following results.

**Theorem 8** Let  $n_0 \geq 2$ ,  $0 < \beta \leq \frac{2}{3}$ . Assume that  $f : \mathbb{D}_R \cup [R, +\infty) \rightarrow \mathbb{C}$  is uniformly continuous and bounded on  $[0, +\infty)$ , is analytic in  $\mathbb{D}_R$ .

(i) If  $0 < \beta < \frac{1}{2}$ ,  $\frac{1}{2} < r < \frac{R}{\max(4q, 2q^2)} < R \leq \frac{1}{2} [n_0]_q^{1-\beta}$ , and  $f$  is not a polynomial of degree  $\leq 1$  in  $\mathbb{D}_R$ , then

$$\|R_{n,q}(f) - f\|_r \sim \frac{1}{[n]_q^\beta}, \quad n \in \mathbb{N}.$$

(ii) If  $\frac{1}{2} < \beta \leq \frac{2}{3}$ ,  $\frac{1}{2} < r < \frac{R}{4q} < R \leq \frac{1}{2} [n_0]_q^{1-\beta}$ , and  $f$  is not a constant function in  $\mathbb{D}_R$ , then

$$\|R_{n,q}(f) - f\|_r \sim \frac{1}{[n]_q^{1-\beta}}, \quad n \in \mathbb{N}.$$

(iii) For  $\beta = \frac{1}{2}$ ,  $\frac{1}{2} < r < \frac{R}{4q^2} < R \leq \frac{1}{2} [n_0]_q^{1-\beta}$ , and  $f$  is not a constant function in  $\mathbb{D}_R$ , then

$$\|R_{n,q}(f) - f\|_r \sim \frac{1}{\sqrt{[n]_q}}, \quad n \in \mathbb{N}.$$

## References

- [1] K. Balázs, Approximation by Bernstein type rational functions, Acta Math. Acad. Sci. Hungar., 26, (1975) 123-134.
- [2] K. Balázs and J. Szabados, Approximation by Bernstein type rational functions, II, Acta Math. Acad. Sci. Hungar., 40 (1982) 3-4, 331-337.
- [3] U. Abel and B. Della Vecchia, Asymptotic approximation by the operators of K. Balázs and Szabados, Acta Sci. Math.(Szeged), 66, (2000) No. 1-2, 137-145.

- [4] O. Dogru, On statistical approximation properties of Stancu type bivariate generalization of  $q$ -Balázs-Szabados operators, Proc. of Int. Conf. on Numer. Anal. and Approx. Th. Cluj-Napoca, Romania, 2002.
- [5] S.G. Gal, Approximation by Complex Bernstein and Convolution Type Operators. World Scientific Publishing, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai (2009)
- [6] N. Ispir and Y. Özkan, Approximation properties of complex  $q$ -Balázs-Szabados operators in compact disks, Journal of Inequalities and Applications 2013, 2013:361
- [7] V. Totik, Saturation for Bernstein-type rational functions, Acta Math. Hungar., 43 (1984), 219–250.
- [8] R. P Agarwal and V. Gupta, On  $q$ -analogue of a complex summation-integral type operators in compact disks, Journal of Inequalities and Applications 2012, 2012:111.
- [9] Andrews G E, Askey R, Roy R. Special functions. Cambridge: Cambridge University Press; 1999.
- [10] S. Ostrovska:  $q$ -Bernstein polynomials and their iterates. J. Approximation Theory 123 (2003), 232–255.
- [11] S. Ostrovska: The sharpness of convergence results for  $q$ -Bernstein polynomials in the case  $q > 1$ . Czech. Math. J. 58 (2008), 1195–1206.
- [12] H. Wang, X. Wu: Saturation of convergence for  $q$ -Bernstein polynomials in the case  $q > 1$ . J. Math. Anal. Appl. 337 (2008), 744–750.
- [13] Z. Wu: The saturation of convergence on the interval  $[0, 1]$  for the  $q$ -Bernstein polynomials in the case  $q > 1$ . J. Math. Anal. Appl. 357 (2009), 137–141.
- [14] N. I. Mahmudov, Approximation by  $q$ -Durrmeyer type polynomials in compact disks in the case  $q > 1$ . Appl. Math. Comput. 237 (2014), 293–303
- [15] N. I. Mahmudov, Approximation by genuine  $q$ -Bernstein-Durrmeyer polynomials in compact disks in the case  $q > 1$ . Abstr. Appl. Anal. 2014, Art. ID 959586, 11 pp.